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On the Rate of Convergence of the Integrated Meyer–König and Zeller Operators for Functions of Bounded Variation

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1. INTRODUCTION

The nth integrated Meyer–König and Zeller operator $\hat{M}_n$, $n \in \mathbf{N}$ (see [1]), associates with a real valued Lebesgue integrable function f defined on $I = [0, 1]$, the function series

$$\hat{M}_n(f, x) = \sum_{k=0}^{\infty} \hat{M}_{nk}(x) \int_{I_k} f(t)\, dt, \tag{1.1}$$

converging for $0 \leqslant x < 1$, with

$$I_k = \left[\frac{k}{n+k}, \frac{k+1}{n+k+1}\right], \qquad k \in \mathbf{N},$$

and

$$\hat{M}_{nk}(x) = (n+1)\binom{n+k+1}{k} x^k (1-x)^n.$$

$\hat{M}_n(f, x)$ can be written as a singular integral of the type

$$\hat{M}_n(f, x) = \int_0^1 H_n(x, t)\, f(t)\, dt, \tag{1.2}$$

with the positive kernel

$$H_n(x, t) = \sum_{k=0}^{\infty} \hat{M}_{nk}(x) \chi_k(t), \tag{1.3}$$

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where χ_k denotes the characteristic function of the interval I_k with respect to I. $\hat{M}_n(f, x)$ is linear, positive, and satisfies

$$\int_0^1 H_n(x, t)\, dt = I. \tag{1.4}$$

The sequence $\{\hat{M}_n : n \in \mathbf{N}\}$ generates a linear approximation method on the normed spaces $L_p(I)$, $1 \leqslant p < \infty$.

R. Bojanic [2] gave the rate of convergence for Fourier series of functions of bounded variation. Fuhua Cheng [3] gave the result of this type for the Bernstein operator. In this paper, using some results of probability theory, we shall give an estimate for the rate of convergence of (1.1) for functions of bounded variation. In the last part, we shall prove that our estimates are essentially the best possible.

2. Lemmas

The proof of our result is based on following lemmas.

Lemma 1. *If $\{\xi_k\}$, $k \in \mathbf{N}$, are independent random variables with the same distribution functions and* $0 < D\xi_k < \infty$, $\beta_3 = E(\xi_r - E\xi_i)^3 < \infty$, *then*

$$\max_y \left| P\left(\frac{1}{b_1 \sqrt{n}}\right) \sum_{k=1}^{n} (\xi_k - a_1) \leqslant y \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt \right| < \frac{c}{\sqrt{n}} \frac{\beta_5}{b_1^3}, \tag{2.1}$$

where $a_1 = E\xi_1$ (expectation of ξ_1), $b_1^2 = D\xi_1 = E(\xi_1 - E\xi_1)^2$ (variance of ξ_1), and $1/\sqrt{2\pi} \leqslant c < 0.82$ (see [4, p. 159]).

Lemma 2. *If $\{\xi_i\}$ $(i = 1, 2, ...)$ are independent random variables with the same geometric distribution*

$$P(\xi_i = k) = x^k(1-x), \qquad 0 < x < 1,\ i = 1, 2, ...,$$

then

$$E\xi_i = \frac{x}{1-x}, \qquad D\xi_i = \frac{x}{(1-x)^2}$$

and $\eta_n = \sum_{i=1}^{n} \xi_i$ is a random variable with distribution

$$P(\eta_n = k) = \binom{n+k-1}{k} x^k (1-x)^n \tag{2.2}$$

(*see* [4, p. 132, Ex. 46, p. 133, Ex. 55, *and* pp. 303, 306]).

LEMMA 3. *For every $x \in (0, 1)$, we have*

$$\left| \sum_{k/(n+k)>x} \binom{n+k-1}{k} x^k(1-x)^n - \frac{1}{2} \right| \leqslant \frac{16x^{-3/2}}{\sqrt{n}}. \tag{2.3}$$

Proof. By Lemma 2, we have

$$\begin{aligned} &\sum_{k/(n+k)>x} \binom{n+k-1}{k} x^k(1-x)^n \\ &\quad = \sum_{k>nx/(1-x)} \binom{n+k-1}{k} x^k(1-x)^n = P\left(\eta_n > \frac{nx}{1-x}\right) \\ &\quad = P\left(\left(\eta_n - \frac{nx}{1-x}\right) \Big/ \sqrt{\frac{nx}{(1-x)^2}} > 0\right). \end{aligned} \tag{2.4}$$

Using Lemma 1 with $a_1 = x/(1-x)$, $b_1 = \sqrt{x/(1-x)^2}$, we have

$$\left| P\left(\frac{\eta_n - nx/(1-x)}{\sqrt{nx}/(1-x)} > 0\right) - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2}\, dt \right| < \frac{\beta_3}{\sqrt{n}\,(\sqrt{x}/(1-x))^3}, \tag{2.5}$$

where

$$\begin{aligned} \beta_3 = E\left|\xi_k - \frac{x}{1-x}\right|^3 &= \sum_{k=0}^{\infty} \left|k - \frac{x}{1-x}\right|^3 x^k(1-x) \\ &\leqslant \sum_{k=0}^{\infty} \left(k^3 + 3k^2 \frac{x}{1-x} + 3k\left(\frac{x}{1-x}\right)^2 + \left(\frac{x}{1-x}\right)^3\right) x^k(1-x). \end{aligned}$$

By an easy calculation we can show that

$$\sum_{k=0}^{\infty} x^k(1-x) = 1, \qquad \sum_{k=0}^{\infty} kx^k(1-x) = \frac{x}{1-x},$$

$$\sum_{k=0}^{\infty} k^2x^k(1-x) = \frac{x(1+x)}{(1-x)^2}, \qquad \sum_{k=0}^{\infty} k^3x^k(1-x) = \frac{x^3+4x^2+x}{(1-x)^3}.$$

Hence

$$\beta_3 \leqslant \frac{x^3+4x^2+x}{(1-x)^3} + \frac{3x}{1-x} \cdot \frac{x(1+x)}{(1-x)^2} + 3\left(\frac{x}{1-x}\right)^3 + \left(\frac{x}{1-x}\right)^3 \leqslant \frac{16}{(1-x)^3}.$$

On the other hand

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2}\, dt = \frac{1}{2}.$$

By (2.5) and (2.6), it follows that

$$\left|\sum_{k/(n+k)>x}\binom{n+k-1}{k}x^k(1-x)^n-\frac{1}{2}\right|\leqslant\frac{1}{\sqrt{n}}\frac{16}{(1-x)^3}\bigg/\left(\frac{\sqrt{x}}{1-x}\right)^3.$$

Inequality (2.3) is proved.

LEMMA 4. *For every* $k\in\mathbf{N}$, $x\in(0,1)$, *we have*

$$\binom{n+k-1}{k}x^k(1-x)^n\leqslant 33/(\sqrt{n}\,x^{3/2}). \tag{2.6}$$

Proof. By (2.2), we have

$$\begin{aligned}\binom{n+k-1}{k}x^k(1-x)^n&=P(\eta_n=k)=P(k-1<\eta_n\leqslant k)\\&=P\left(\frac{k-1-nx/(1-x)}{\sqrt{nx}/(1-x)}<\frac{\eta_n-nx/(1-x)}{\sqrt{nx}/(1-x)}\right.\\&\left.\leqslant\frac{k-nx/(1-x)}{\sqrt{nx}/(1-x)}\right).\end{aligned}$$

Using the method of proof of Lemma 3, we can show that

$$\left|P(\eta_n=k)-\frac{1}{\sqrt{2\pi}}\int_{k-1-nx/(1-x)/\sqrt{nx}/(1-x)}^{k-nx/(1-x)/\sqrt{nx}/(1-x)}e^{-t^2/2}\,dt\right|\leqslant\frac{32}{\sqrt{n}\,x^{3/2}}.$$

But the absolute value of the second term on the left-hand side of the last inequality is less than $(1-x)/\sqrt{2\pi nx}$; hence

$$P(\eta_n=k)\leqslant\frac{33}{\sqrt{n}\,x^{3/2}}.$$

Lemma 4 is proved.

LEMMA 5. *For* $x\in(0,1)$, $n\in\mathbf{N}$, *we have*

$$\hat{M}_n((t-x)^2,x)=\frac{x(1-x)^2}{n}+o_x\left(\frac{1}{n}\right). \tag{2.7}$$

Proof. First

$$\hat{M}_n((t-x)^2, x) = (n+1)(1-x)^n \sum_{k=0}^{\infty} \binom{n+k+1}{k} x^k \int_{I_k} (t-x)^2 \, dt$$

$$= (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \cdot \frac{1}{3}\left[\left(\frac{k+1}{n+k+1} - x\right)^2 + \left(\frac{k+1}{n+k+1} - x\right)\left(\frac{k}{n+k} - x\right) + \left(\frac{k}{n+k} - x\right)^2\right]$$

$$= (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \left[\frac{1}{3}\left(\frac{k+1}{n+k+1} - \frac{k}{n+k}\right)^2 + \left(\frac{k+1}{n+k+1} - x\right)\left(\frac{k}{n+k} - x\right)\right]$$

$$= (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \left[\frac{1}{3}\left(\frac{n}{(n+k+1)(n+k)}\right)^2 + \left(\frac{k+1}{n+k+1} - \frac{k}{n+k}\right)\left(\frac{k}{n+k} - x\right) + \left(\frac{k}{n+k} - x\right)^2\right].$$

Using [5, (5.9.2)—(5.9.7)], we obtain

$$(1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \cdot \frac{1}{3}\left(\frac{n}{(n+k+1)(n+k)}\right)^2 = o_x\left(\frac{1}{n}\right),$$

$$(1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \left(\frac{k}{n+k} - x\right)^2 = \frac{x(1-x)^2}{n-1} - o_x\left(\frac{1}{n}\right),$$

$$(1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \left(\frac{k+1}{n+k+1} - \frac{k}{n+k}\right)\left(\frac{k}{n+k} - x\right)$$

$$= (1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \left[\frac{1}{n+k+1}\frac{k}{n+k} - \frac{1}{n+k+1} \times \left(\frac{k}{n+k}\right)^2 - \frac{x}{n+k+1} + \frac{x}{n+k+1}\frac{k}{n+k}\right] = o_x\left(\frac{1}{n}\right).$$

Consequently (2.7) is proved.

LEMMA 6. *If n is sufficiently large, $x \in (0, 1)$, then*:

(1) *For $0 \leqslant y < x$, we have*

$$\int_0^y H_n(x, t) \, dt \leqslant \frac{2x(1-x)^2}{n(x-y)^2}. \tag{2.8}$$

(2) *For* $x < z \leqslant 1$, *we have*

$$\int_z^1 H_n(x, t)\, dt \leqslant \frac{2x(1-x)^2}{n(z-x)^2}. \tag{2.9}$$

Proof. First, by (2.7), for n sufficiently large, we have

$$\frac{x(1-x)^2}{2n} \leqslant \hat{M}_n((t-x)^2, x) \leqslant \frac{2x(1-x)^2}{n}. \tag{2.10}$$

If $0 \leqslant y < x$, $t \in [0, y]$, then

$$\frac{x-t}{x-y} \geqslant 1.$$

By (2.10), for n sufficiently large, we have

$$\begin{aligned}\int_0^y H_n(x, t)\, dt &\leqslant \int_0^y \left(\frac{x-t}{x-y}\right)^2 H_n(x, t)\, dt \\ &\leqslant \frac{1}{(x-y)^2} \int_0^1 (x-t)^2 H_n(x\ t)\, dt \\ &= \frac{1}{(x-y)^2} \hat{M}_n((t-x)^2, x) \leqslant \frac{2x(1-x)^2}{n(x-y)^2}.\end{aligned}$$

Inequality (2.8) is proved. The proof of (2.9) is similar.

LEMMA 7. *For* $n \geqslant 2$, $0 \leqslant x \leqslant 1$, *and* $m \geqslant 1$, *we have*

$$\hat{M}_n((t-x)^{2m}, x) \leqslant A_m n^{-m}, \tag{2.11}$$

where A_m *is independent of* n *and* x (see [1, (2.1)]).

3. MAIN THEOREM

THEOREM. *Let* f *be a function of bounded variation on* $[0, 1]$ *and let* $\bigvee_a^b(g_x)$ *be the total variation of* g_x *on* $[a, b]$. *Then, for every* $x \in (0, 1)$ *and* n *sufficiently large, we have*

$$\begin{aligned}&|\hat{M}_n(f, x) - \tfrac{1}{2}(f(x+) + f(x-))| \\ &\quad \leqslant \frac{5}{nx} \sum_{k=1}^{n} \bigvee_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}} (g_x) + \frac{49}{\sqrt{n}\, x^{3/2}} |f(x+) - f(x-)|,\end{aligned} \tag{3.1}$$

where

$$g_x(t)=\begin{cases} f(t)-f(x+), & x<t\leqslant 1; \\ 0, & t=x; \\ f(t)-f(x-), & 0\leqslant t<x. \end{cases}$$

Proof. First

$$\begin{aligned}|\hat{M}_n(f, x)-\tfrac{1}{2}(f(x+)+f(x-))| \\ \leqslant |\hat{M}_n(g_x, x)+\tfrac{1}{2}|f(x+)-f(x-)|\,|\hat{M}_n(\text{sign}(t-x), x)|. \end{aligned} \tag{3.2}$$

This shows that to estimate $|\hat{M}_n(f, x)-\frac{1}{2}(f(x+)+f(x-))|$ we only have to evaluate $\hat{M}_n(g_x, x)$ and $\hat{M}_n(\text{sign}(t-x), x)$.

We first estimate $\hat{M}_n(\text{sign}(t-x), x)$. If $k'/(n+k')\leqslant x<(k'+1)/(n+k'+1)$, then

$$\begin{aligned}&\hat{M}_n(\text{sign}(t-x), x)\\ &= \sum_{k/(n+k)>x} M_{nk}(x)-\sum_{(k+1)/(n+k+1)<x} M_{nk}(x)\\ &\quad+(n+1)\left[\int_x^{(k'+1)/(n+k'+1)} dt-\int_{k'/(n+k')}^{x} dt\right]\hat{M}_{nk'}(x)\\ &\overset{\text{def}}{=} A_n(x)-B_n(x)+C_n(x),\end{aligned}$$

where $M_{nk}(x)=\binom{n+k-1}{k}x^k(1-x)^n$. By Lemma 3, we have

$$\left|A_n(x)-\frac{1}{2}\right|\leqslant\frac{16}{\sqrt{n}\,x^{3/2}}.$$

But $|C_n(x)|\leqslant M_{nk'}(x)$, so by Lemma 4, we have

$$|C_n(x)|\leqslant\frac{33}{\sqrt{n}\,x^{3/2}}.$$

Since $B_n(x)=|-A_n(x)-M_{nk'}(x)|$, we have

$$\left|B_n(x)-\frac{1}{2}\right|\leqslant\left|A_n(x)-\frac{1}{2}\right|+M_{nk'}(x)\leqslant\frac{49}{\sqrt{n}\,x^{3/2}}.$$

Consequently

$$|\hat{M}_n(\text{sign}(t-x), x)|\leqslant\frac{98}{\sqrt{n}\,x^{3/2}}. \tag{3.3}$$

To estimate $\hat{M}_n(g_x, x)$, we decompose $[0, 1]$ into

$$\hat{I}_1 = \left[0, x - \frac{x}{\sqrt{n}}\right], \qquad \hat{I}_2 = \left[x - \frac{x}{\sqrt{n}}, x + \frac{1-x}{\sqrt{n}}\right], \qquad \hat{I}_3 = \left[x + \frac{1-x}{\sqrt{n}}, 1\right].$$

Using (1.2), we have

$$\begin{aligned}\hat{M}_n(g_x, x) &= \int_0^1 g_x(t)\, H_n(x, t)\, dt \\ &= \left(\int_{\hat{I}_1} + \int_{\hat{I}_2} + \int_{\hat{I}_3}\right) g_x(t)\, H_n(x, t)\, dt \\ &\overset{\text{def}}{=} \Delta_{1,n}(f, x) + \Delta_{2,n}(f, x) + \Delta_{3,n}(f, x).\end{aligned}$$

Let$\lambda_n(x, t) = \int_0^t H_n(x, u)\, du$.

First, we evaluate $\Delta_{2,n}(f, x)$. For $t \in \hat{I}_2$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leqslant \bigvee_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}} (g_x)$$

and so

$$|\Delta_{2,n}(f, x)| \leqslant \bigvee_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}} (g_x) \int_{\hat{I}_2} H_n(x, t)\, dt \leqslant \bigvee_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}} (g_x). \tag{3.4}$$

Since the evaluation of $\Delta_{1,n}(f, x)$ is similar to work in [3], we shall omit some details. Let $y = x - x/\sqrt{n}$. Using partial Lebesgue–Stieltjes integration, we find that

$$\begin{aligned}|\Delta_{1,n}(f, x)| &= \left|\int_0^y g_x(t)\, d_t \lambda_n(x, t)\right| \\ &= \left|g_x(y+)\, \lambda_n(x, y) - \int_0^y \lambda_n(x, t)\, d_t\, g_x(t)\right| \\ &\leqslant \bigvee_{y+}^{x} (g_x)\, \lambda_n(x, y) + \int_0^y \lambda_n(x, t)\, d_t\left(-\bigvee_t^x (g_x)\right).\end{aligned}$$

By (2.8), we have

$$|\Delta_{1,n}(f, x)| \leqslant \bigvee_{y+}^{x} (g_x) \frac{2x(1-x)^2}{n(x-y)^2} + \frac{2x(1-x)^2}{n} \int_0^y \frac{1}{(x-t)^2}\, d_t\left(-\bigvee_t^x (g_x)\right).$$

Furthermore, since

$$\int_0^y \frac{1}{(x-t)^2}\, d_t\left(-\bigvee_t^x (g_x)\right) = -\frac{\bigvee_{y+}^x (g_x)}{(x-y)^2} + \frac{\bigvee_0^x (g_x)}{x^2} + 2\int_0^y \bigvee_t^x (g_x) \frac{dt}{(x-t)^3},$$

we have

$$|\Delta_{1,n}(f,x)| \leqslant \frac{2x(1-x)^2}{n}\left(\frac{\bigvee_0^x(g_x)}{x^2} + 2\int_0^{x-x/\sqrt{n}} \bigvee_t^x(g_x)\frac{dt}{(x-t)^3}\right).$$

Replacing the variable t in the last integral by $x - x/\sqrt{t}$, we find that

$$\int_0^{x-x/\sqrt{n}} \bigvee_t^x(g_x)\frac{2dt}{(x-t)^3} = \frac{1}{x^2}\int_1^n \bigvee_{x-x/\sqrt{n}}^x (g_x)\,dt \leqslant \frac{1}{x^2}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_x).$$

Hence

$$|\Delta_{1,n}(f,x)| \leqslant \frac{2x(1-x)^2}{nx^2}\left(\bigvee_0^x(g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_x)\right)$$
$$\leqslant \frac{4(1-x)^2}{nx}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_x). \tag{3.5}$$

Using a similar method and (2.9), we obtain

$$|\Delta_{3,n}(f,x)| \leqslant \frac{4x}{n}\sum_{k=1}^n \bigvee_x^{x+(1-x)/\sqrt{k}} (g_x). \tag{3.6}$$

From (3.4)–(3.6) it follows that

$$|\hat{M}_n(g_x, x)| \leqslant \frac{5}{nx}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x). \tag{3.7}$$

The theorem now follows from (3.3) and (3.7).

4. Remark

We shall prove that our estimate is essentially the best possible. Consider the function $f(t) = |t-x|$ $(0<x<1)$ on $[0, 1]$. By (2.10), for any small δ and n sufficiently large, we have

$$\hat{M}_n(|t-x|, x) = \int_0^1 |t-x|\, H_n(x,t)\,dt$$
$$= \left(\int_{x-\delta}^{x+\delta} + \int_{|t-x|>\delta}\right)|t-x|\, H_n(x,t)\,dt$$
$$\leqslant \delta + \frac{1}{\delta}\int_0^1 (t-x)^2 H_n(x,t)\,dt \leqslant \delta + \frac{2x(1-x)^2}{n\delta} \tag{4.1}$$

and

$$\hat{M}_n(|t-x|, x) \geqslant \int_{x-\delta}^{x+\delta} |t-x| \, H_n(x, t) \, dt$$
$$\geqslant \frac{1}{\delta} \int_{x-\delta}^{x+\delta} (t-x)^2 \, H_n(x, t) \, dt$$
$$= \frac{1}{\delta} \left[\int_0^1 - \int_{|t-x| > \delta} \right] (t-x)^2 \, H_n(x, t) \, dt$$
$$\geqslant \frac{x(1-x)^2}{2n\delta} - \frac{1}{\delta} \int_{|t-x| > \delta} (t-x)^2 \, H_n(x, t) \, dt.$$

Using (2.11), we have

$$\int_{|t-x| > \delta} (t-x)^2 \, H_n(x, t) \, dt \leqslant \frac{1}{\delta^2} \hat{M}_n((t-x)^4, x) \leqslant \frac{A_2}{\delta^2 n^2},$$

where A_2 is a constant. Hence

$$\hat{M}_n(|t-x|, x) \geqslant \frac{x(1-x)^2}{2n\delta} - \frac{A_2}{n^2 \delta^3}. \tag{4.2}$$

Choosing $\delta = 2\sqrt{A_2/nx(1-x)^2}$, we obtain from (4.1) and (4.2) that

$$\frac{x^{3/2}(1-x)^3}{8\sqrt{A_2 n}} \leqslant \hat{M}_n(|t-x|, x) \leqslant \frac{(2A_2+1)(x(1-x)^2)^{-1/2}}{\sqrt{nA_2}}. \tag{4.3}$$

On the other hand, from (3.1), since $\bigvee_{x-\beta}^{x+\alpha}(f) = \alpha + \beta$, it follows that

$$|\hat{M}_n(f, x) - f(x)| \leqslant \frac{5}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (f)$$
$$\leqslant \frac{5}{nx} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leqslant \frac{5}{\sqrt{n}\, x}. \tag{4.4}$$

Hence, by comparing (4.3) and (4.4), we see that (3.1) cannot be asymptotically improved for a function of bounded variation.

Finally, we mention that, using the method of this paper, we can obtain other, similar results on Meyer–König and Zeller operators.

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